

COMPARISON OF FOUR NUMERICAL METHODS FOR FLOOD ROUTING^a

Discussion by Danny L. Fread,² M. ASCE

The author has presented a valuable comparison of four finite difference techniques for solving the complete one-dimensional Saint Venant equations of unsteady flow. Since the writer has found the four-point implicit method to offer important advantages as mentioned by the author, the following comments are concerned with this particular technique.

The author states that the four-point implicit method cannot be used if there is no rating equation or hydrograph available at the downstream boundary. When there is no important flow disturbance downstream of the routing reach which can propagate into the reach and influence the flow, a rating equation is always available. Such a rating equation is the Chezy or Manning equation which provides a single-valued or multivalued stage-discharge relationship. The extent to which the relationship is multivalued, as manifested by the "loop rating curve," is dependent upon the extent to which the ratio, S_f/S_o , departs from unity. The four-point implicit method, which utilizes the generalized Newton iteration technique to solve the system of nonlinear finite difference equations, is well suited for using a specified downstream boundary condition formulated from the Chezy or Manning equation. Such a boundary condition is given by the following expression written in terms of the Chezy equation:

$$Q_j^{n+1} - (CAR^{1/2} S_f^{1/2})_j^{n+1} = 0 \dots\dots\dots (28)$$

in which S_f^{n+1} is approximated from Eq. 2 expressed in finite difference form for the Δx_{j-1} subreach.

The author points out a potential difficulty in selecting a proper time step when applying the four-point implicit method to floods that exceed the channel capacity and propagate along the overbank or flood plain. As the author states, the rate of propagation is different for the channel flow as compared to the overbank flow due to differences in the hydraulic properties of each. In addition, when the channel meanders through the flood plain, the time of travel is different through each due to differences in the two reach lengths. The one-dimensional Saint Venant equations can be formulated to conveniently simulate flows in meandering channels with flood plains as follows.

Let $Q = Q_c + Q_f$, where the subscripts c and f denote channel and flood plain, respectively. Also, assume that Q_c and Q_f are related by means of the Chezy equation in which S_f is approximated by the water surface slope, $\Delta h/\Delta x$. Then, the Saint Venant equations become:

^aJuly, 1974, by Roland K. Price (Proc. Paper 10659).

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$$\frac{\partial(K_1 Q)}{\partial x_c} + \frac{\partial(K_2 Q)}{\partial x_f} + \frac{\partial(A_c + A_f)}{\partial t} = 0 \dots \dots \dots (29)$$

$$\frac{\partial Q}{\partial t} + \frac{\partial\left(\frac{K_1^2 Q^2}{A_c}\right)}{\partial x_c} + \frac{\partial\left(\frac{K_2^2 Q^2}{A_f}\right)}{\partial x_f} + gA_c\left(\frac{\partial h}{\partial x_c} + S_{fc}\right) + gA_f\left(\frac{\partial h}{\partial x_f} + S_{ff}\right) = 0 \dots \dots \dots (30)$$

$$\text{in which } K_1 = \frac{1}{1 + K} \dots \dots \dots (31)$$

$$K_2 = \frac{K}{1 + K} \dots \dots \dots (32)$$

$$K = \frac{C_f A_f}{C_c A_c} \left(\frac{R_f \Delta x_c}{R_c \Delta x_f} \right)^{1/2} = \frac{Q_f}{Q_c} \dots \dots \dots (33)$$

and h = the water surface elevation such that $\partial h / \partial x = \partial y / \partial x - S_o$.

When the four-point implicit method is applied to Eqs. 29 and 30 to determine the unknowns, Q and h , the coefficient matrix in the generalized Newton iteration technique has exactly the same form as when Eqs. 1 and 2 are used. This feature of Eqs. 29 and 30 is an important convenience.

The selection of an optimum time step suitable for both the channel and flood-plain flows can be accomplished by means of the selection of the proper size space step, Δx_f , for the flood plain. Since

$$\Delta t = \frac{\Delta x_c}{C_c} \dots \dots \dots (34)$$

$$\text{and } \Delta t = \frac{\Delta x_f}{C_f} \dots \dots \dots (35)$$

$$\text{then } \Delta x_f = \left(\frac{C_f}{C_c} \right) \Delta x_c \dots \dots \dots (36)$$

Thus, if Δx_c is selected according to spatial variations in channel geometry and to provide a suitably large ratio (wave length/ Δx_c) for desirable convergence properties, then Eq. 36 will provide guidance in the selection of Δx_f . In this way Δt will be optimum for both the channel and flood-plain flows. Since C_f is less than C_c , Δx_f will be smaller than Δx_c ; this should present no difficulty since the spatial variations in the geometry of the flood plain are usually less than for the channel.

Although the author states that he encountered no instability when the 9 weighting factor of the four-point implicit method was 1/2, the writer has experienced the weakly stable condition associated with θ of 1/2 when simulating some floods in the Lower Mississippi River. The weakly stable condition is manifested by bounded oscillations of the solution about the true solution. The

writer found that they could be eliminated by either increasing θ to about 0.55 or decreasing Δt from 24 hr to about 3 hr.

The writer (19) has investigated the stability of the four-point implicit method for the following simplified version of Eqs. 1 and 2:

$$\frac{\partial h}{\partial t} + Y_o \frac{\partial v}{\partial x} = 0 \quad \dots \dots \dots (37)$$

$$\frac{\partial v}{\partial t} + g \frac{\partial h}{\partial x} + kv = 0 \quad \dots \dots \dots (38)$$

$$\text{in which } k = \frac{2gV_o}{C^2 Y_o} \quad \dots \dots \dots (39)$$

h = the water surface elevation; and Y_o and V_o are mean values of depth and velocity, respectively. An expression for stability (in the sense of the von Neumann conjecture that linear operators with variable coefficients are stable if all their localized operators in which the coefficients are taken constant are stable) is given by the following expression:

$$|\lambda| = \left[\frac{1 + (2\theta - 2)^2 a + (\theta - 1)b}{1 + 4\theta^2 a + \theta b} \right]^{1/2} \quad \dots \dots \dots (40)$$

in which $a = gY_o(\Delta t/\Delta x)^2 \tan^2(\pi\Delta x/L)$; $b = k\Delta t$; and L = wavelength = wave celerity \times duration.

If $|\lambda| < 1$, independent of the values of Δx and Δt , the errors due to truncation and roundoff will not grow with time, and the difference equations are unconditionally linearly stable. This is the case when $1/2 \leq \theta \leq 1$, although only weakly stable (i.e., $|\lambda| = 1$) when $\theta = 1/2$ and k approaches zero.

Appendix.—Reference

19. Fread, D. L., "Numerical Properties of Implicit Four-Point Finite Difference Equations of Unsteady Flow," *NOAA Technical Memo NWS Hydro-18*, U.S. Department of Commerce, National Oceanic and Atmospheric Administration, National Weather Service, 1974.